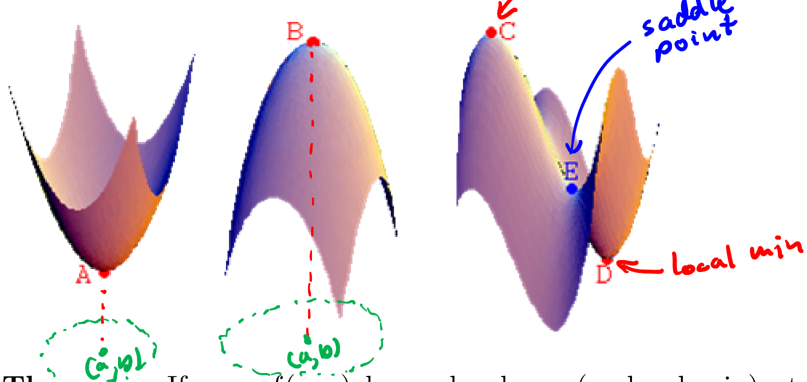
Sec 14.7: Maximum and Minimum Values

DEF. Let $z = f(x, y)$ be defined on a region planar $\mathcal{R}$. Suppose $(a, b) \in \mathcal{R}$. Then:
1. $f(a, b)$ is a local minimum value of f if $f(x, y) \geq f(a, b)$ for all points (x, y) in an open disk centered at (a, b).

2. $f(a, b)$ is a local maximum value of f if $f(x, y) \leq f(a, b)$ for all points (x, y) in an open disk centered at (a, b).



Theorem. If $z = f(x, y)$ has a local max(or local min) at the point (a, b), and both partial derivatives at the point (a, b) exist, then

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0.$$

Critical Point. Is an interior point (a, b) in the domain of the function where either

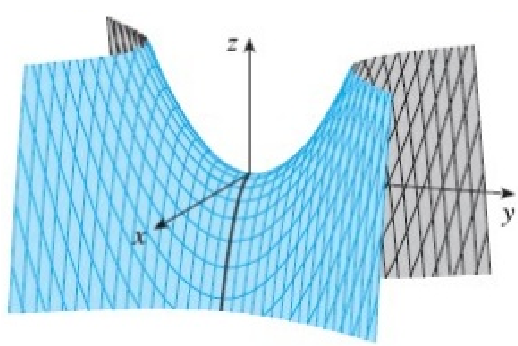
$$\begin{cases} f_x(a, b) = 0 \\ f_y(a, b) = 0 \end{cases}$$

or where one or both $f_x(a, b)$ and $f_y(a, b)$ do not exist.

Saddle Point

$z = y^2 - x^2$

$\begin{cases} -2x = 0 \\ 2y = 0 \end{cases} \Rightarrow \begin{matrix} x = 0 \\ y = 0 \end{matrix}$



Let $f(x, y) = y^2 - x^2$. Since $f_x = 2x$ and $f_y = 2y$, the only critical point is $(0, 0)$. Notice that for points on the x-axis we have $y = 0$, so $f(x, y) = -x^2 < 0$ (if $x \neq 0$). However, for points on the y-axis we have $x = 0$, so $f(x, y) = y^2 > 0$ (if $y \neq 0$). Thus every disk with center $(0, 0)$ contains points where f takes positive values as well as points where f takes negative values. Therefore, $f(0, 0) = 0$ cannot be a local max nor local min.This motivates the following definition.

A function $f(x, y)$ has a **saddle point** at a critical point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points (x, y) where $f(x, y) < f(a, b)$.
In this case, the graph of $z = f(x, y)$, nearby the saddle point looks like a Pringle's potato chip.

Ex1. Find all critical points of $P(x,y) = x^3 - 12xy + 8y^3$.

$P_x = 3x^2 - 12y$; $P_y = -12x + 24y^2$

set $\begin{cases} 3x^2 - 12y = 0 \\ -12x + 24y^2 = 0 \end{cases} \Rightarrow \begin{cases} x^2 - 4y = 0 \ \dots (1) \\ -x + 2y^2 = 0 \ \dots (2). \end{cases}$

From (2): $x = 2y^2 \dots (*)$

Replace $(*)$ in (1): $(2y^2) - 4y = 0$

$4y^4 - 4y = 0 \Rightarrow 4y(y^3 - 1) = 0$

$\Rightarrow y = 0,\ y = 1$

∘) when $y = 0$, $x = 2(0)^2 = 0$

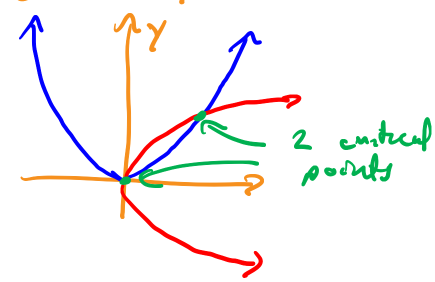
∘) when $y = 1$, $x = 2(1)^2 = 2$

so the critical points are $(0,0)$ and $(2,1)$.

Extra Notes:

From (1): $x^2 = 4y$

From (2): $x^2 = 2y^2$



2^{nd} Derivative Test. Suppose $f(x,y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a,b) and that $f_x(a,b) = f_y(a,b) = 0$. Let D be the quantity defined by

$$D := f_{xx}(a,b) \cdot f_{yy}(a,b) - \Big[f_{xy}(a,b)\Big]^2$$

Then, we have the following.

1) If $D > 0$ and $f_{xx}(a,b) > 0$, f has a local min. at (a,b). concave up

2) If $D > 0$ and $f_{xx}(a,b) < 0$, f has a local max. at (a,b). concave down

3) If $D < 0$, f has a saddle point at (a,b).

4) If $D = 0$, no conclusion can be drawn.

Ex2. Find and classify the critical point(s) of the function $P(x,y) = x^3 - 12xy + 8y^3$.

From Ex 1: the critical points are $(0,0)$ and $(2,1)$.

tools: $P_{xx} = 6x$; $P_{yy} = 48y$; $P_{xy} = -12 = P_{yx}$

$\boxed{\text{C.P. } (0,0)}$

$D = P_{xx}(0,0) \cdot P_{yy}(0,0) - [P_{xy}(0,0)]^2 = (0)(0) - [-12]^2 < 0$

since $D < 0$, there is a saddle point at $(0,0)$ by the Second Derivative Test

$\boxed{\text{C.P. } (2,1)}$

$D = P_{xx}(2,1) \cdot P_{yy}(2,1) - [P_{xy}(2,1)]^2 = (12)(48) - (-12)^2$

$= 12(48 - 12) > 0$

since $D > 0$, we check $P_{xx}(2,1) = 12 > 0$.

By the Second Derivative Test there is a local min. at $(2,1)$.

Exercises.
(1) Find and classify all critical points of the function $g(x, y) = x^2y + 4xy + 4y^2$.
(2) Find all critical points of $Q(x, y) = (x^2 + y^2) \exp(y^2 - x^2)$.

Sec 14.7 <u>Absolute</u> Maxima and Minima on closed, bounded regions.

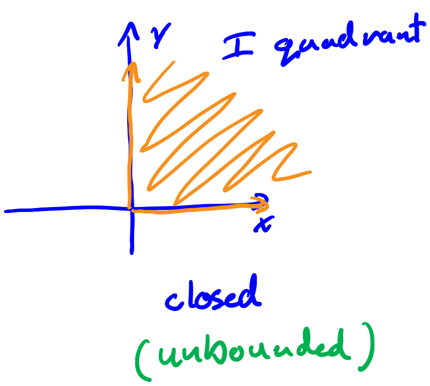
Absolute Maximum and Minimum Values:

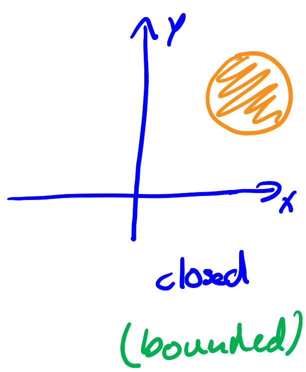
Let (a, b) be a point in the domain D of a function f of two variables. Then $f(a, b)$ is the

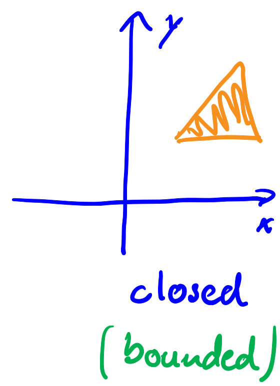
- **absolute maximum value** of f on D if $f(a, b) \geq f(x, y)$ for all (x, y) in D.
- **absolute minimum value** of f on D if $f(a, b) \leq f(x, y)$ for all (x, y) in D.

Closed and Bounded Regions:

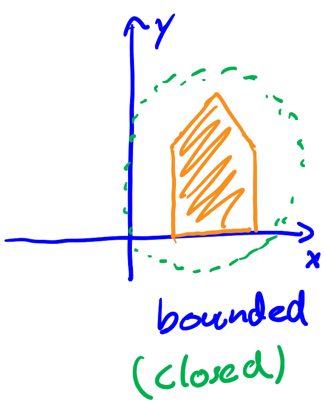
- A **closed region** in $\mathbb{R}^2$ is a set that contains all its boundary points.

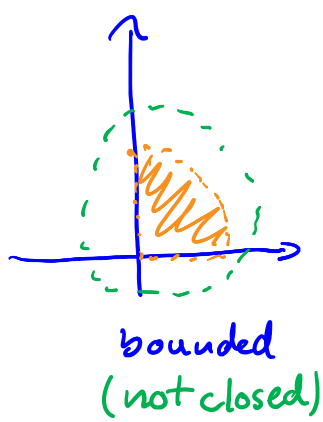




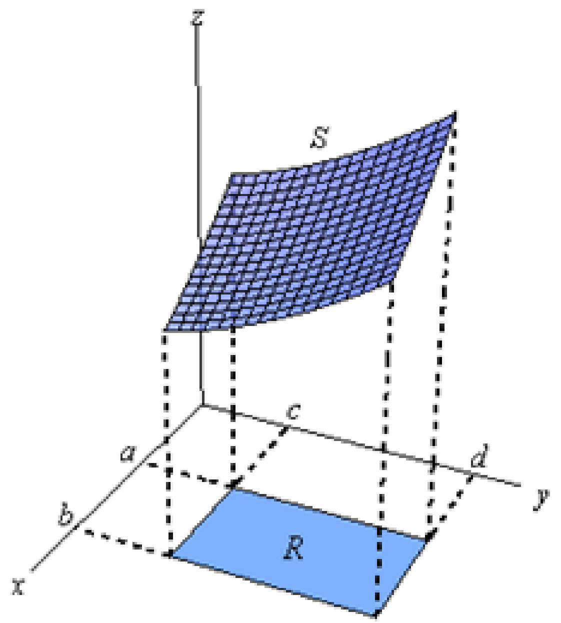


- A **bounded region** in $\mathbb{R}^2$ is a set that is contained within some disk.





Thm: Extreme Value Theorem Let $z = f(x, y)$ be a continuous function over the region $\mathcal{R}$ in $\mathbb{R}^2$. If $\mathcal{R}$ is closed and bounded, then f attains an absolute maximum and an absolute minimum over the region $\mathcal{R}$.



Algorithm: To find the absolute maximum and minimum values of a continuous function on a closed, bounded region $\mathcal{R}$, do the following steps:

Step 1 : List the interior critical points and evaluate f at these points.

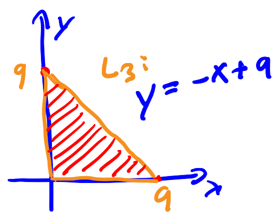
Step 2 : List the boundary points where f may have local maxima and minima and evaluate f at these points.

Step 3 : The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Ex3. Find the absolute maximum and the absolute minimum values of the function

$$f(x,y) = 2 + 2x + 4y - x^2 - y^2$$

on the closed triangular region $\mathcal{R}$ with vertices $(0,0)$, $(0,9)$ and $(9,0)$.



f is continuous on R, and R is closed and bounded. By the Extreme Value thm, f attains an ABSOLUTE MAX. and ABSOLUTE min over R.

Critical pts in the interior:

$f_x = 2-2x$
$f_y = 4-2y$ $\Rightarrow$ $\begin{cases} 2-2x \overset{\text{set}}{=} 0 \Rightarrow x=1 \\ 2-2y=0 \Rightarrow y=2 \end{cases}$

the only critical pt is $(2,1)$ (it is in the region)

Boundary

L_1: $x=0$, $0 \le y \le 9$
$f(0,y) = 2+4y-y^2$
Let $g(y) = 2+4y-y^2$, $0 \le y \le 9$
then $g'(y) = 4-2y$
Set derivative equal to zero
$4-2y=0 \Rightarrow y=2$
one canidate: $(0,2)$
other canidates: $(0,0)$, $(0,9)$

L_2: $y=0$, $0 \le x \le 9$
$f(x,0) = 2+2x-x^2$
Let $h(x) = 2+2x-x^2$, $0 \le x \le 9$
then $h'(x) = 2-2x$
set derivative equal to zero
$2-2x=0 \Rightarrow x=1$
one canidate: $(1,0)$
other canidates: $(0,1)$, $(9,0)$

L_3: $y=-x+9$, $0 \le x \le 9$
$f(x, 9-x) = 2+2x+4(9-x)-x^2-(9-x)^2$
Let $M(x) = 2+2x+36-4x-x^2-(x^2-18x+81)$
then $M'(x) = 2-4-2x-2x+18$
set $16-4x=0 \Rightarrow x=4$
one canidate: $(4,5)$
other canidates: $(0,9)$, $(9,0)$

Canidates

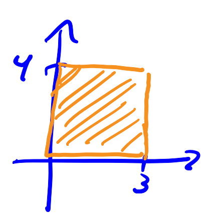
	(a,b)	$f(a,b) = 2+2a+4b-a^2-b^2$
Interior C.P.	$(1,2)$	$f(1,2) = 7$
boundary	$(0,2)$	$f(0,2) = 6$
	$(1,0)$	$f(1,0) = 3$
	$(4,5)$	$f(4,5) = -11$
corners	$(0,0)$	$f(0,0) = 2$
	$(0,9)$	$f(0,9) = -61$
	$(9,0)$	$f(9,0) = -43$

The absolute maximum value is $\boxed{7}$

The absolute minimum value is $\boxed{-61}$

Exercise.
Find the absolute maximum value and absolute minimum value of $f(x,y) = xy - x - 2y$ on the region $D = \{(x,y) \in \mathbb{R}^2 : \ 0 \le x \le 3, \ |y-2| \le 2\}$.

$-2 \le y-2 \le 2$
$0 \le y \le 4$

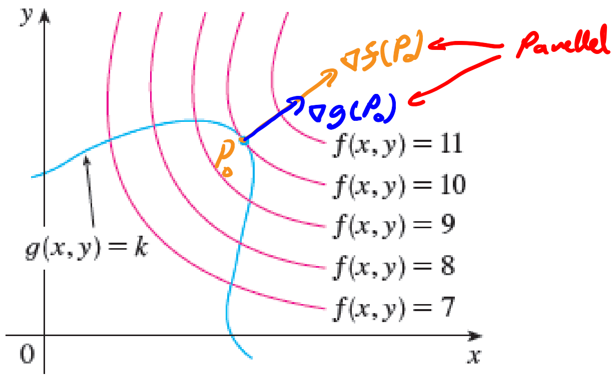


Sec 14.8: Lagrange Multipliers

Suppose $f(x,y)$ and $g(x,y)$ are differentiable functions. Let C be the level curve defined by the equation $g(x,y)=k$. If $P_0=(a,b)$ is a point on the curve C for which $f(P_0)$ is the absolute maximum (or minimum) of $f(x,y)$ along the curve C, then $\nabla f(P_0)$ and $\nabla g(P_0)$ must be parallel; that is

$$\nabla f(P_0) = \lambda \nabla g(P_0)$$

for some real number λ.



Method of Lagrange Multipliers (2 variables)[This method assumes that the extreme values exist and $\nabla g \neq \mathbf{0}$ on the curve $g(x,y)=k$]. To find the maximum and minimum values of $f(x,y)$ subject to the constraint $g(x,y)=k$ we do the following:

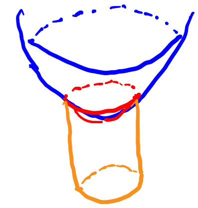
(a) Find all values of x, y and λ such that

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = k \end{cases}$$

(b) Evaluate f at all points (x,y) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Ex1. What are the extreme values of the function $f(x,y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$?

max / min



max/ min $\quad f(x,y) = x^2 + 2y^2$

subject to $\quad \underbrace{x^2+y^2}_{g(x,y)} = 1$

we will use Lagrange Multipliers:

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 1 \end{cases} \Rightarrow \begin{cases} \boxed{\begin{array}{l} \langle 2x, 4y \rangle = \lambda \langle 2x, 4y \rangle \\ x^2+y^2=1 \end{array}} \end{cases}$$

"main equations" (*)

then

$$\begin{cases} 2x = 2\lambda x \\ 4y = 2\lambda y \\ x^2+y^2=1 \end{cases} \Rightarrow \begin{cases} x(1-\lambda) = 0 & (i) \\ y(2-\lambda) = 0 & (ii) \\ x^2+y^2=1 & (iii) \end{cases}$$

From (i) $\boxed{x=0}$ $\boxed{\lambda=1}$

Case I: $x=0$ $\quad \begin{cases} y(2-\lambda)=0 & (ii) \\ x^2+y^2=1 & (iii) \end{cases}$

From (iii): $0^2+y^2=1 \Rightarrow y=1$ or $y=-1$

Canidates: $(0,1)$, $(0,-1)$

Case II: $\lambda=1$ $\quad \begin{cases} y(2-\lambda)=0 & (ii) \\ x^2+y^2=1 & (iii) \end{cases}$

In (ii): $y(2-1)=0 \Rightarrow y=0$

then in (iii): $x^2+0^2=1 \Rightarrow x=1, x=-1$

canidates: $(1,0)$, $(-1,0)$

Canidates

(a,b)	$f(a,b) = a^2+b^2$
$(0,1)$	2
$(0,-1)$	2
$(1,0)$	1
$(-1,0)$	1

the ABS. MAX. value is 2, it occurs at $(0,1)$ and $(0,-1)$.

the ABS. MIN. value is 1, it occurs at $(1,0)$ and $(-1,0)$.

Extra Notes

$\nabla f(0,-1) \,//\, \nabla g(0,-1)$? $\quad g(0,-1)=1$?

$$\begin{cases} \langle 2(0), 4(-1) \rangle = \lambda \langle 2(0), 2(-1) \rangle \Rightarrow \langle 0,-4 \rangle = \lambda \langle 0,-2 \rangle \checkmark \\ 0^2+(-1)^2=1 \quad \checkmark \end{cases}$$

Ex2. Use Lagrange multipliers to find the maximum and minimum values of the function

$$f(x,y) = x^2 + x + 2y^2$$

over the planar region $x^2 + y^2 \leq 1$.

Let $g(x,y) = x^2+y^2$

•) when $x^2+y^2<1$:

$f_x = 2x+1$
$f_y = 4y$
$\Rightarrow$ set $\begin{cases} 2x+1=0 \Rightarrow x = -\frac{1}{2} \\ 4y=0 \Rightarrow y=0 \end{cases}$

critical pt is $(-\frac{1}{2}, 0)$
(It satisfies $(-\frac{1}{2})^2 + 0^2 < 1$)

•) when $x^2+y^2=1$ (when using Lagrange multipliers)

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 1 \end{cases} \Rightarrow \boxed{\begin{cases} \overbrace{\langle 2x+1, 4y\rangle}^{\langle f_x, f_y\rangle} = \lambda \overbrace{\langle 2x, 2y\rangle}^{\langle g_x, g_y\rangle} \\ x^2+y^2=1 \end{cases}} \quad (*)$$

then $\begin{cases} 2x+1 = 2\lambda x \\ 4y = 2\lambda y \\ x^2+y^2=1 \end{cases} \Rightarrow \begin{cases} 2x+1 = 2\lambda x & (i) \\ 2y(2-\lambda)=0 & (ii) \\ x^2+y^2=1 & (iii) \end{cases}$

From (ii) $y=0$ or $\lambda=2$

case I $y=0$ $\begin{cases} 2x+1 = 2\lambda x & (i) \\ x^2+y^2=1 & (iii) \end{cases}$

using (iii): $x^2+0^2=1 \Rightarrow x=1, x=-1$
canidates: $(1,0), (-1,0)$

case II $\lambda=2$ $\begin{cases} 2x+1 = 2\lambda x & (i) \\ x^2+y^2=1 & (iii) \end{cases}$

using (i): $2x+1 = 2(2)x \Rightarrow 1 = 2x \Rightarrow x = \frac{1}{2}$
In (iii): $(\frac{1}{2})^2 + y^2 = 1 \Rightarrow y^2 = \frac{3}{4} \Rightarrow y = \frac{\sqrt{3}}{2}, y = -\frac{\sqrt{3}}{2}$.
Candidates: $(\frac{1}{2}, \frac{\sqrt{3}}{2}), (\frac{1}{2}, \frac{-\sqrt{3}}{2})$.

Canidates

(a,b)	$f(a,b) = a^2+a+2b^2$
$(-\frac{1}{2}, 0)$	$-\frac{1}{4}$
$(1,0)$	2
$(-1,0)$	0
$(\frac{1}{2}, \frac{\sqrt{3}}{2})$	$\frac{9}{4}$
$(\frac{1}{2}, -\frac{\sqrt{3}}{2})$	$\frac{9}{4}$

the ABS. MAX. value is $\boxed{9/4}$

the ABS. MIN. value is $\boxed{-\frac{1}{4}}$

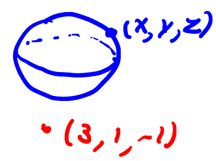
Method of Lagrange Multipliers (3 variables). To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$:

(a) Find all values of x, y, z, and λ such that

$$\begin{cases} \nabla f(x, y, z) = \lambda \nabla g(x, y, z) \\ g(x, y, z) = k \end{cases}$$

(b) Evaluate f at all points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Ex3. Find the point(s) on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to the point $(3, 1, -1)$.



$$\begin{cases} \text{minimize distance} = \sqrt{(x+3)^2+(y-1)^2+(z+1)^2} \\ \text{subject to } x^2+y^2+z^2=4. \end{cases}$$

then we rewrite

minimize $\text{distance}^2 = f(x,y,z) = (x+3)^2+(y-1)^2+(z+1)^3$

subject to $\underbrace{x^2+y^2+z^2}_{g(x,y,z)}=4$

we will use Lagrange's Method

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) \end{cases} \Rightarrow \begin{cases} \langle 2(x-3), 2(y-1), 2(z+1)\rangle = \lambda(2x, 2y, 2z) \\ x^2+y^2+z^2=4 \end{cases}$$

then $\begin{cases} 2(x-3)=2\lambda x \\ 2(y-1)=2\lambda y \\ 2(z+1)=2\lambda z \\ x^2+y^2+z^2=4 \end{cases} \Rightarrow \begin{cases} x-3=\lambda x \ (i) \\ y-1=\lambda y \ (ii) \\ z+1=\lambda z \ (iii) \\ x^2+y^2+z^2=4 \ (iv) \end{cases}$

Note: If $\lambda=1$, in (i): $x-3=x \Rightarrow -3=0$ "Impossible", so $\lambda \neq 1$.

From (i), (ii), (iii): we get $x = \frac{3}{1-\lambda} = 3\left(\frac{1}{1-\lambda}\right)$; $y = \frac{1}{1-\lambda}$; $z = -\frac{1}{1-\lambda}$

In (iv): $\frac{9}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} = 4 \Rightarrow \frac{11}{(1-\lambda)^2} = 4 \Rightarrow \frac{1}{(1-\lambda)^2} = \frac{4}{11}$

so, $\frac{1}{1-\lambda} = \pm\sqrt{\frac{4}{11}}$

-) when $\frac{1}{1-\lambda} = +\sqrt{\frac{4}{11}}$, we get $x = 3\left(\sqrt{\frac{4}{11}}\right)$, $y=\sqrt{\frac{4}{11}}$, $z = -\left(\sqrt{\frac{4}{11}}\right)$

-) when $\frac{1}{1-\lambda} = -\sqrt{\frac{4}{11}}$, we get $x = 3\left(-\sqrt{\frac{4}{11}}\right)$, $y = -\sqrt{\frac{4}{11}}$, $z = \sqrt{\frac{4}{11}}$

we get the candidates $\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, \frac{-2}{\sqrt{11}}\right)$ and $\left(\frac{-6}{\sqrt{11}}, \frac{-2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$.

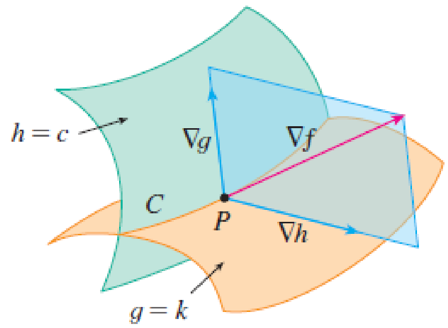
The closest point is $\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, \frac{-2}{\sqrt{11}}\right)$.

Exercise. A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

Lagrange Multipliers with two constraints

Suppose now that we want to find the maximum and minimum values of $f(x, y, z)$ subject to the constraints $g(x, y, z) = k$ and $h(x, y, z) = c$. In this case we need to find all values of x, y, z, λ and μ such that

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) \\ g(x,y,z) = k \\ h(x,y,z) = c \end{cases}$$

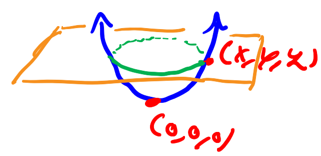


Ex4. The plane $x + y + 2z = 12$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. **Use Lagrange multipliers with two constraints** to find the points on the ellipse that are nearest to and farthest from the origin.

maximize/minimize distance$^2 = f(x,y,z) = x^2+y^2+z^2$

subject to $x+y+2z = 12$

$x^2+y^2-z = 0$



let $g(x,y,z) = x+y+2z$ and $h(x,y,z) = x^2+y^2-z$.

(*)

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) \\ g(x,y,z) = 12 \\ h(x,y,z) = 0 \end{cases} \Rightarrow \boxed{\begin{cases} \langle 2x, 2y, 2z\rangle = \lambda\langle 1,1,2\rangle + \mu\langle 2x, 2y, -1\rangle \\ x+y+2z=12 \\ x^2+y^2-z=0 \end{cases}}$$

$$\begin{cases} 2x = \lambda + \mu x & (i) \\ 2y = \lambda + 2\mu y & (ii) \\ 2z = 2\lambda - \mu & (iii) \\ x+y+2z = 12 & (iv) \\ x^2+y^2-z=0 & (v) \end{cases}$$

IF $\mu = 1$: in (i): $\cancel{2x} = \lambda + \cancel{2x} \Rightarrow \lambda = 0$

in (iii): $2z = 0 - 1 \Rightarrow z = -\frac{1}{2}$

in (v): $x^2+y^2 = z \Rightarrow \underbrace{x^2+y^2 = -\frac{1}{2}}_{\text{False}}$

"contradiction"

so $\mu \neq 1$

From (i) and (ii): $2x(1-\mu) = \lambda$ and $2y(1-\mu) = \lambda$

$\Rightarrow 2x(1-\mu) = 2y(1-\mu) \Rightarrow \boxed{x=y}$

From (iv): $x+x+2z=12 \Rightarrow 2x+2z = 12 \Rightarrow \boxed{x = 6-z}$ or $\boxed{z = 6-x}$

In (v): $x^2+x^2-(6-x) = 0 \Rightarrow 2x^2 + x - 6 = 0 \Rightarrow x = \dfrac{-1 \pm \sqrt{1-4(2)(-6)}}{2(2)}$

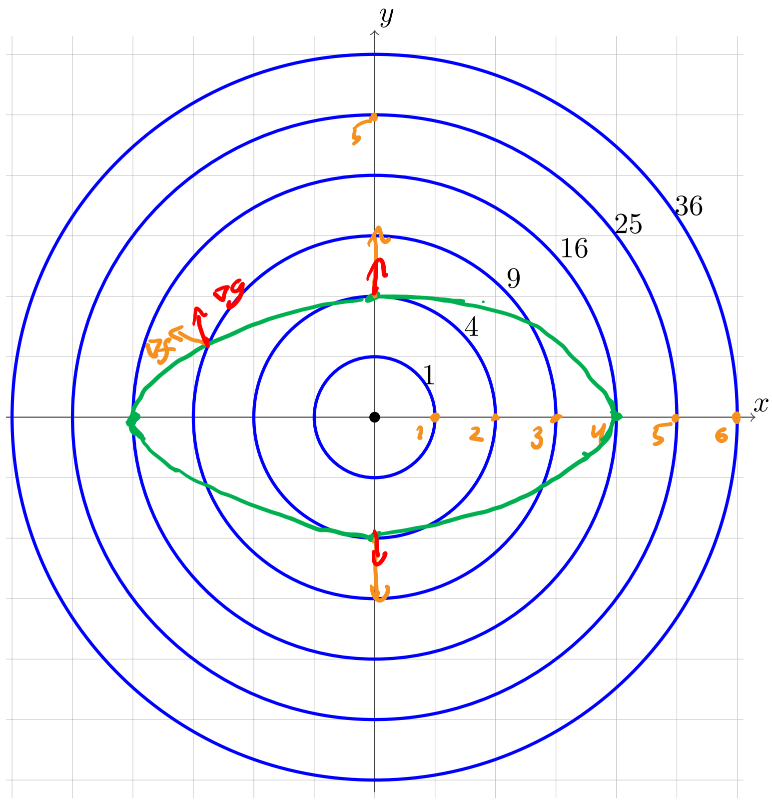
$\Rightarrow x = \dfrac{-1 \pm 7}{4}$

when $x = \frac{6}{4} = \frac{3}{2}$, we get $(x,y,z) = \left(\frac{3}{2}, \frac{3}{2}, \frac{9}{2}\right)$ (Nearest point)

when $x = \frac{-8}{4} = -2$, we get $(x,y,z) = (-2,-2,8)$. (Farthest point)

Exercise. Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

Ex5. Sketch the curve $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$ on the figure below.
Note: circles represent some level curves of the function $f(x, y) = x^2 + y^2$.



max/min $f(x,y) = x^2 + y^2$
subject to $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$

$$\boxed{\begin{array}{l} \nabla f(x,y) = \lambda \nabla g(x,y) \\ \dfrac{x^2}{16} + \dfrac{y^2}{4} = 1 \end{array}}$$

We want to identify the absolute maximum value and the absolute minimum value of the function $f(x, y) = x^2 + y^2$ subject to the contraint $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$.

Use the picture to complete the following:

- The candidates (a, b) for the location of absolute extrema using the method of Lagrange Multipliers are:

  $(4,0)$, $(-4,0)$, $(0,2)$, $(0,-2)$

- The absolute maximun value is: 16

- The absolute minimun value is: 4